

MODELING ROTATING STARS IN TWO DIMENSIONS

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Abstract. In this lecture I present the way stars can be modeled in two dimensions and especially the fluid flows that are driven by rotation. I discuss some of the various ways of taking into account turbulence and conclude this contribution by a short presentation of some of the first results obtained with the ESTER code on the modeling of interferometrically observed fast rotating early-type stars.

1 Introduction

Rotation remains, with magnetic fields, an ill-known quantity in the interiors of stars. It is however associated with fascinating objects like massive stars or the first stars of the Universe. These latter objects are indeed often called the factories of metals and because of their compactness (due to low opacity) are usually thought to have been fast rotators. However, understanding the mechanisms that lead to the enrichments of the interstellar medium with the variety of elements, requires the understanding of the mechanisms which mix the stars or which simply bring the elements from the regions of their nucleosynthesis to the surface where they can be expelled by winds.

The determination of the mean flows that pervade rotating stars is therefore an unavoidable step towards this understanding. This is why much work has been devoted to insert the effects of flows into one dimensional models (e.g. Maeder & Meynet 2000; Maeder 2009). Even if this modeling succeeded in explaining a variety of effects of rotation (e.g. the ratio of blue to red supergiants in galaxies or the abundance of lithium), 1D models cannot include many specificities of rotating fluid flows. Indeed, a fluid flow is intrinsically a multidimensional phenomenon. Two dimensions of space are therefore the minimum number of dimensions to compute a rather general flow. For instance, the geostrophic flow of rotating fluids, that comes from by the domination of the Coriolis force, verifies the

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Taylor-Proudman theorem stating that $\frac{\partial \vec{\rho} \vec{v}}{\partial z} = \vec{0}$. Such a condition is not compatible with the existence of a stellar core. In fact the very problem is that rotation imposes a cylindrical symmetry while gravity imposes spherical symmetry. The combination of both yields a two dimensional problem.

In the following I therefore propose to focus our attention on the mean flows that pervade a rotating star. We will thus have the opportunity to go through the processes (baroclinicity and Reynolds stresses) that drive a secular mixing of the stars. Then, we will shortly present the state of the ESTER project and some selected first results on the modeling of early-type fast rotating stars.

2 The problem's formulation

In order to make a first step into the 2D-modeling of rotating stars, we shall consider an isolated, non-magnetic, not-mass-losing, early-type star. Moreover, we shall forget about any time evolution: the star is powered by nuclear reactions but these do not influence the chemical composition. However, there are some regions with turbulent flows. There we assume that the turbulence is in a statistically steady state. Such an ideal star does not exist but some stars like Vega may be close to it.

With all the foregoing precautions, we may formulate the problem of a 2D-model of a rotating star in a consistent way.

The partial differential equations that govern the steady-state of our ideal rotating star read:

$$\left\{ \begin{array}{l} \Delta \phi = 4\pi G \langle \rho \rangle \\ \langle \rho T \vec{v} \cdot \vec{\nabla} S \rangle = -\text{div} \vec{F} + \varepsilon_* \\ \langle \rho \vec{v} \cdot \vec{\nabla} \vec{v} \rangle = -\vec{\nabla} \langle P \rangle - \langle \rho \rangle \vec{\nabla} \phi + \vec{F}_v \\ \text{div} \langle \rho \vec{v} \rangle = 0. \end{array} \right. \quad (2.1)$$

These are Poisson equation for the gravitational potential ϕ , the energy equation involving the temperature T , the density ρ , the entropy S , the diffusive heat flux \vec{F} and the nuclear heat sources ε_* , the momentum equation with the viscous force \vec{F}_v and the equation of mass conservation. The brackets $\langle \dots \rangle$ indicate time averages. They are necessary to remove turbulence fluctuations. Actually, all quantities are time averaged.

Equations (2.1) should be completed by the prescriptions of the microphysics, namely the equation of state, the opacities, the nuclear network etc. They should also be supplemented with boundary conditions. All these are discussed in Espinosa Lara & Rieutord (2013). Here, we shall only focus on the problems raised by the velocity and temperature fields boundary conditions. Before that we need to concentrate on the mean flows.

3 The mean flows in rotating stars

The mean flows that pervade a rotating star and enforce some mixing have various origin. The most common sources are the following:

- baroclinicity
- Reynolds stresses
- gravitational contraction, mass loss or mass accretion

Let us review these phenomena in more details.

3.1 Baroclinicity

In fluid mechanics baroclinicity refers to the inclination of isobars and isotherms (or isopycnic, equipotential surfaces...). This is indeed a natural feature of rotating stars that was discovered long ago by von Zeipel (1924). Since much confusion has emerged after this work on the origin of the meridional circulation, I'd like to say a few words about stellar baroclinic flows in order to clarify the phenomenon and the way it should be approached.

Baroclinicity emerges in fluids because pressure and temperature obey two different and independent equations: there are some coupling through buoyancy and heat advection but these are weak. Hence, if we think to these two fields (pressure and temperature), they live independently and therefore establish their own system of isosurfaces. Usually, they do not coincide and baroclinicity arises. It arises because density depends on temperature or on both temperature and pressure. Hence, isodensity surfaces are distinct from isobars or, in other words, the two vectors $\vec{\nabla}\rho$ and $\vec{\nabla}P$ are not parallel. When taking the curl of the momentum equation (after division by ρ), one gets

$$M(\vec{v}) = \frac{\vec{\nabla}P \times \vec{\nabla}\rho}{\rho^2}$$

where the right hand side is usually called the baroclinic torque and M is a differential operator. The consequence of this torque is that a flow arises. In stars this flow is basically a differential rotation, namely a purely azimuthal velocity field. However, as viscosity is always present, a differential rotation, like any shear flow, transport momentum in the direction of the velocity gradients. In a steady state the momentum flux induced by viscosity must be compensated by some meridional flow. Such a balance is illustrated by the φ component of the momentum equation, which may be written

$$v_s \partial_s (s v_\varphi) + v_z \partial_z (s v_\varphi) = \nu s \Delta' v_\varphi$$

or, in a more condensed form,

$$\vec{v} \cdot \vec{\nabla} (s^2 \Omega) = \nu \vec{\nabla} \cdot (s^2 \vec{\nabla} \Omega) \quad (3.1)$$

where we used the cylindrical coordinates (s, φ, z) and where Δ' represents some Laplacian-like operator.

As we may note on (3.1), advection of angular momentum $\vec{v} \cdot \vec{\nabla}(s^2\Omega)$ just compensates the viscous force due to angular velocity gradients $\nu \vec{\nabla} \cdot (s^2 \vec{\nabla} \Omega)$. In axisymmetric situations like that of our rotating stars, advection is that of meridional circulation. We therefore see that in a steady state meridional circulation is controlled only by viscosity.

Confusion arose in the past because it was assumed that rotation generates a thermal disequilibrium. Indeed, in a barotropic fluid

$$\text{div}(\chi \vec{\nabla} T) \neq 0$$

known as von Zeipel's paradox. The paradox was solved by saying that the meridional circulation velocity \vec{v} was such that

$$\rho T \vec{v} \cdot \vec{\nabla} S = \text{div}(\chi \vec{\nabla} T) \quad \text{and} \quad \text{div}(\rho \vec{v}) = 0$$

namely, heat advection by meridional circulation compensates the thermal imbalance and verifies mass conservation. *Such a reasoning is incorrect* because the driving of a flow is due to forces or torques, not to mass and energy conservation which cannot ensure angular momentum conservation. As shown by Busse (1981), a thermal imbalance leads to a time-evolution of the temperature field itself, which is more rapid than a mechanical rearrangement of the fluid. The foregoing discussion is detailed in Rieutord (2006b).

3.2 Reynolds stresses

3.2.1 The scale of turbulence

Baroclinic flows therefore slowly advect heat, chemicals and angular momentum. However, the differential rotation of baroclinic origin is a mere shear flow that may develop instabilities if the Reynolds number is large enough or if the Richardson number is low enough. These instabilities usually lead to turbulence. Unlike thermal convection where the linear instability drives motion at scales on the fraction of the radius, shear instabilities in a stably stratified fluid inject energy at small-scale. To see that, we recall that the Richardson criterion taking into account the high thermal diffusivity of the fluid imposes (Zahn 1992)

$$\frac{N^2}{S_h^2} \frac{v\ell}{\kappa} \lesssim 1/4$$

where N is the Brunt-Väisälä frequency, S_h the local shear, v and ℓ the velocities and length scale of the eddies and κ the heat diffusivity. This criterion originally proposed by Townsend (1958), says that shear instability grows up to a scale (of the eddies) where heat diffusion is unable to smooth out the stable buoyancy of the background. Since the largest eddies have a turn-over time scale $1/S_h$, we may observe that all the eddies forced by the background shear are such that

$$\frac{N^2}{S_h^2} \frac{\ell^2}{\tau} \lesssim \kappa/4, \quad \text{with} \quad \tau \lesssim S_h^{-1}$$

or

$$\frac{\ell}{R} \lesssim \left(\frac{\kappa S_h}{4R^2 N^2} \right)^{1/2}$$

where we introduced R the radius of the star. Let us put numbers in this expression. From Espinosa Lara & Rieutord (2013), we evaluate the associated turbulent kinematic viscosity

$$\nu_t = \frac{S_h^2 \kappa}{12N^2} \sim 10^6 \text{ cm}^2/\text{s}$$

The global shear of differential rotation is of a fraction of the rotation rate; let us write $S_h = f\Omega$. Hence we find

$$\frac{\ell}{R} \lesssim \left(\frac{\nu_t}{4fR^2\Omega} \right)^{1/2} = \frac{\sqrt{E}}{\sqrt{2f}}$$

where E is the Ekman number based on the shear induced turbulence. Since $E < 10^{-8}$ and $f \sim 0.05$ (see Espinosa Lara & Rieutord 2013), we find that

$$\ell/R \lesssim 3 \times 10^{-4}$$

Hence, turbulence in vertically stratified region remains on relatively small-scales in the vertical direction. In the horizontal directions of course the scales can be large (but limited by the stability of the eddies).

As far as convection zones are concerned, nothing prevents the instability from driving all the scales where the Schwarzschild criterion predicts convection. Hence these are fully mixed.

3.2.2 Reynolds stresses

The convection zones are fully mixed regions, but because of the strong turbulence they harbour, Reynolds stresses drive mean flows like the differential rotation of the Sun. These stresses and the associated flows also force some flows in the neighbouring radiative zones. The picture that emerges from these remarks is that turbulence is everywhere in stars, with variable intensity, properties, and a modeling of its effects is required. Here we first concentrate on its momentum transport that is on Reynolds stresses. If we use Reynolds decomposition, namely

$$\vec{v} = \langle \vec{v} \rangle + \vec{v}'$$

where \vec{v}' are the velocity fluctuations with respect to the average, the Reynolds stress tensor is

$$\mathbf{R} = \langle \rho \vec{v}' \otimes \vec{v}' \rangle$$

Usually, correlations with density are not important (a fortiori in an incompressible fluid!) and the components of the Reynolds stress tensor are often written

$$R_{ij} = \langle v'_i v'_j \rangle$$

In a steady state, and neglecting correlations with density, mean flows therefore verify:

$$\rho \langle v_j \rangle \partial_j \langle v_i \rangle + \partial_j (\rho \langle v'_i v'_j \rangle) = -\partial_i \langle P \rangle + \rho g_i + F_i$$

where \vec{F} is the viscous force.

The main question with the previous equation is the expression of the Reynolds stress tensor as a function of the mean-field, the so-called closure problem of mean field equations.

A basic way of solving this problem is to assume that turbulent stresses are like fluid stresses and that they can be represented by a turbulent viscosity. We thus may write

$$\rho \langle v_j \rangle \partial_j \langle v_i \rangle = -\partial_i \langle P \rangle + \rho g_i + F_i^T$$

with

$$\begin{aligned} \vec{F}^T = & \mu_t \left[\Delta \vec{v} + \frac{1}{3} \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) + 2 (\vec{\nabla} \ln \mu_t \cdot \vec{\nabla}) \vec{v} \right. \\ & \left. + \vec{\nabla} \ln \mu_t \times (\vec{\nabla} \times \vec{v}) - \frac{2}{3} (\vec{\nabla} \cdot \vec{v}) \vec{\nabla} \ln \mu_t \right]. \end{aligned} \quad (3.2)$$

where μ_t is the turbulent dynamical viscosity of the gas. This expression enables local variations of the viscosity although it does not say anything on its determination. But there is a more fundamental difficulty: there is no reason why the functional form of the Reynolds stress should be

$$\langle \rho v'_i v'_j \rangle = -\mu_t (\partial_i \langle v_j \rangle + \partial_j \langle v_i \rangle - \frac{2}{3} \partial_k \langle v_k \rangle \delta_{ij}) \quad (3.3)$$

as expected for a newtonian fluid. The analogy of turbulence with such a fluid is very limited because this expression is derived from the assumption that fluid flows are slight perturbations of the thermodynamic equilibrium. It is unclear what is the statistical equilibrium of turbulence and how actual flows may slightly deviates from it. Moreover, turbulence particles are vortices that support long range interactions, not collisions...

Now, if we still admit (3.3), the expression of μ_t is still a problem. In the nineteenthies, Prandtl introduced the idea of the mixing length suggesting that in turbulent shear flows

$$\mu_t = \rho \Lambda^2 \left| \frac{\partial v_x}{\partial z} \right|$$

where Λ is the mixing length and z the direction of the shear. This assumption leads to interesting results on wall-turbulence, but misses the point on the axis of a turbulent jet (where it predicts zero-viscosity!).

Another recipe is the Smagorinski's prescription saying that $\mu_t = \rho \Delta^2 |\langle c_{ij} \rangle|$ where $\langle c_{ij} \rangle$ is the mean shear. Such a prescription is more general and still often used in Large-Eddy Simulations (LES) as a basic subgrid model. However, as the mixing-length model, this is a local prescription that does not take care of non-local effects.

To avoid this pitfall, more sophisticated models have been developed for engineering problems. One of the most popular is the $K-\varepsilon$ model of Launder & Spalding (1972): this model adds two new equations for the evolution of K the turbulent kinetic energy and ε the energy dissipation. The advection diffusion equation that controls the evolution of these quantities are derived from the evolution of second-order correlations with a simple model of the third-order correlation (Rieutord 1997). The point is that when K and ε are known, the turbulent pressure and the turbulent viscosity are also known. These models may be useful to determine statistically steady inhomogeneous flows that we find in stars. However, because of the huge increase of the computing power, these models have been forsaken to the profit of LES, which are more flexible for industrial flows. In astrophysics, this raises the problem of the comparison with observations, which cannot be as detailed as laboratory experiments.

Another line of research that should be mentioned, is the one followed by Rüdiger and collaborators (see Rüdiger 1989; Kitchatinov et al. 1994). This is the mean-field approach of turbulence where one seeks for an expression of $\langle v'_i v'_j \rangle$ and other second order correlations, as functions of large-scale quantities like local rotation rate. In this approach the Reynolds tensor is not reduced to that of the functional form of a newtonian fluid. New effects like the Λ -effect or anisotropic turbulent viscosities, appear. These approaches of course raise new kinds of problems (see Snellman et al. 2012).

3.3 Gravitational contraction, mass accretion and mass-loss

The last source of large-scale motions in a rotating star is due to expansion or contraction of the star at various phase of its evolution. In non-rotating stars these phenomena just lead to radial flows, but in a rotating star, angular momentum conservation leads to a differential rotation and an associated meridional circulation. Not much is known on these flows that are currently under investigations (Hypolite & Rieutord in preparation). Again, the associated shear will lead to some small-scale turbulence in radiative regions.

3.4 Conclusion

To conclude the foregoing discussion, let us underline that we considered here only non-magnetic processes. Magnetic fields will of course complicate the picture. An open question is how all these sources compete together and of course how we should model the associated turbulence.

4 Heat transport

4.1 convection zones

The foregoing discussion focused on turbulent viscosity and more generally on momentum transport. However, heat transport was really the first obstacle to circumvent when first stellar models were constructed. The well-known mixing-length theory, based on Prandtl's idea, offers a plausible answer to this problem. In this approach, all the difficulties of turbulent heat transport are condensed in a dimensionless parameter of order unity α_{MLT} that has been calibrated on the Sun. Unfortunately, such a parameter is not universal and varies from star to star, impeding any precise prediction. A clear example is given by the two stars of α Cen (Eggenberger et al. 2004).

In two dimensions, the difficulty is squared. First because the 1D-MLT has no straightforward generalisation to 2D. Turbulent transport is now also in latitude via turbulent diffusion and mean flows.

A first idea is use the “down-gradient” prescription for the heat flux (this is a prescription that is largely used in the $K - \varepsilon$ model). Here, since thermal convection disappears when the entropy gradient vanishes or becomes positive (in the direction opposite to effective gravity), a natural way to model the convective heat flux is to assume that it is proportional to the entropy gradient, namely

$$\vec{F}_{\text{conv}} = \langle \rho c_p T \vec{v} \rangle = -\chi_{\text{turb}} T \vec{\nabla} S / \mathcal{R}$$

where \mathcal{R} is the ideal gas constant.

When reduced to 1D this prescription can be related to the classical MLT. It may therefore be viewed as a generalization of this theory. However, there are other possibilities like the introduction of an eddy conductivity χ_t such that $\vec{F} = -\chi_t \vec{\nabla} T$ (Rüdiger 1989).

4.2 The surface heat transfer

One of the new problem raised by 2D-models concerns the implementation of the boundary conditions that should be met by the temperature field. A simple prescription that replaces a detailed atmospheric model is to impose that the star radiates locally as a black body, namely that

$$-\chi_r \vec{n} \cdot \vec{\nabla} T = \sigma T^4 \quad (4.1)$$

where χ_r is the radiative conductivity, σ Stefan-Boltzmann constant and \vec{n} the outer normal of the surface. In non-rotating models this condition is usually applied where the optical depth is $\tau_s = 2/3$, but $\tau_s = 1$ is also used (Morel 1997). In 2D-models, the situation is more complicated: one has to define the surface where boundary conditions (for velocity, gravitational potential and temperature) are imposed. Since the opacity is a rapidly varying quantity at the surface of a star an iso-optical-depth surface is not convenient numerically. We therefore chose to impose boundary conditions on an isobar and select the isobar that “emerges” at the pole, i.e. whose pressure is

$$P_s = \tau_s \frac{g_{\text{pole}}}{\kappa_{\text{pole}}}, \quad (4.2)$$

On this isobar $T = T_{\text{eff}}$ at the pole only. As one moves towards the equator this isobar sinks into the optically thick matter because at $\tau_s = 2/3$ pressure is stronger at the pole than at the equator (recall that gravity is larger at the pole). The trick here is to determine the temperature on this isobar as a function of the latitude. For this we assume that the matter lying above this isobar behaves like a polytrope of index n . With this hypothesis, it can be shown (cf Espinosa Lara & Rieutord 2013) that the temperature on this isobar follows the law

$$T_b(\theta) = \left(\frac{g_{\text{pole}}}{g_{\text{eff}}(\theta)} \frac{\kappa(\theta)}{\kappa_{\text{pole}}} \right)^{1/(n+1)} \left(\frac{-\chi_r \vec{n} \cdot \vec{\nabla} T}{\sigma} \right)^{1/4} \quad (4.3)$$

which fully determines the temperature field. In more sophisticated models, $T_b(\theta)$ may be determined by the modeling of the stellar atmosphere.

5 Computing the baroclinic flows in 2D-models

We focus on early-type stars so as to avoid the modeling of outer convective layers. Thermal convection is only modeled in the core of the star. One-dimensional models tell us that it is very efficient and therefore the assumption of an isentropic core is quite good (Maeder 2009). ESTER models therefore assume isentropic cores and no further modeling of core turbulence is included yet.

5.1 Flows in the radiative envelope

Above the core, the radiative envelope is not in hydrostatic equilibrium as recalled above. Baroclinic flows pervade it and are responsible of the mixing. They represent the first difficulty for the computation of self-consistent models. When determined, we have a prediction of the differential rotation and the meridional circulation.

As we noted in sect. 3.1 the determination of these flows rests on the fluid’s viscosity, which is crucial for stationary solutions. The importance of viscosity is shown by the value of the Ekman number, namely

$$E = \frac{\nu}{2\Omega R^2}$$

As shown in Espinosa Lara & Rieutord (2013), its value is always less than 10^{-8} even if shear turbulence is accounted for.

Viscous effects are therefore expected to be small. But they are crucial as they lift the degeneracy of the differential rotation as we shall show now.

Let us take the curl of the momentum equation and let us project it in the azimuthal direction on \vec{e}_φ . If viscous or Reynolds stresses are neglected then one finds:

$$s \frac{\partial \Omega^2}{\partial z} = \frac{\vec{\nabla} P \times \vec{\nabla} \rho}{\rho^2} \cdot \vec{e}_\varphi \quad (5.1)$$

The solution of (5.1) is invariant with respect to the addition of an arbitrary function of s ; if $\Omega(s, z)$ is a solution

$$\Omega'^2(s, z) = \Omega^2(s, z) + F(s)$$

is also a solution. As shown in Rieutord (2006a), viscosity lifts this kind of degeneracy. This may be understood in the following way. The differential rotation depends on $F(s)$. The balance of angular momentum advection and diffusion is given by (3.1) which is completed by mass conservation. Once Ω is known, these two equations give the meridional circulation, but without matching the boundary conditions. In general, the derived meridian flow does not verify

$$\vec{v} \cdot \vec{n} = 0$$

on the stellar surface. This condition is enforced by an Ekman boundary layer where the mass flux into the surface is identified with the mass pumping of the boundary layer. Indeed, the Ekman boundary layer completes an inviscid solution like $s\Omega(s, z) + G(s)$ in such a way that stress-free conditions are met, but the boundary layer correction $\tilde{u}_\theta, \tilde{u}_\varphi$ usually do not verify mass conservation. This latter constraint is corrected by the velocity component orthogonal to the layer, which is called the pumping. The identification of the pumping of the layer with the mass flux generated by the circulation at the stellar surface gives the differential equation for the unknown geostrophic flow that appears in inviscid solutions ($F(s)$ or $G(s)$). This differential equation is derived in Rieutord (2006a), in the simplified case of an incompressible “star” not far from rigid rotation.

In modeling rotating stars in 2D we are not interested in the boundary layer because they are very thin and would require a high spatial resolution near the surface. This is why Espinosa Lara & Rieutord (2013) derived a special boundary condition that completes (5.1) and insure stress-free conditions without an explicit computation of the boundary layer flow. This boundary condition reads

$$E_s s^2 \vec{\xi} \cdot \vec{\nabla} \Omega + \psi \vec{\tau} \cdot \vec{\nabla} (s^2 \Omega) = 0 \quad \text{at the surface.} \quad (5.2)$$

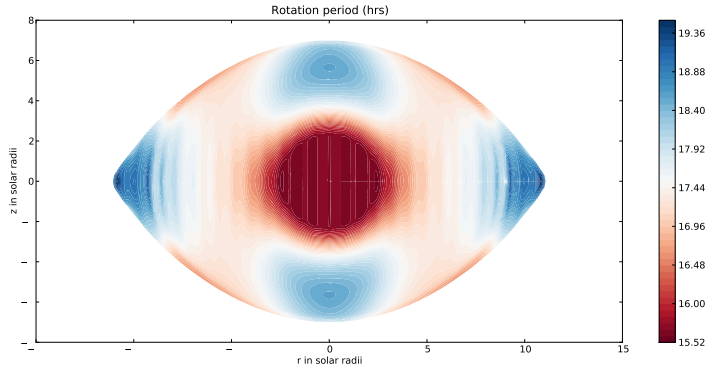


Fig. 1. Meridional view of the differential rotation of $30 M_{\odot}$ ZAMS star rotating at 98% of its critical angular velocity with $X=0.7$ and $Z=0.02$.

where E_s is the surface Ekman number, $\vec{\xi}$ and $\vec{\tau}$ unit vectors normal and tangential to the surface, while ψ is the stream function of the meridional circulation. It turns out that this circulation scales as the interior Ekman number, therefore this small parameter drops out of the momentum equation and only order one quantities are computed.

5.2 The Core-Envelope Interface

The foregoing simplification unfortunately breaks down at the core-envelope interface (CEI). Indeed, as shown by Espinosa Lara & Rieutord (2013) the chemical evolution of the convective core, that leads to a density discontinuity at the CEI, also leads to an angular velocity discontinuity on this interface. In addition we may also consider that turbulent viscosity steeply increases when one goes from the envelope to the core. These discontinuities are triggering the so-called Stewartson layer that develops along the tangential cylinder circumventing the core. This layer may play a crucial role in the chemical enrichment of the envelope by products of the nuclear reactions. Its computation is however demanding as it scales like $E^{1/4}$ but fortunately less than the Ekman layer whose thickness scales like $E^{1/2}$. More studies are necessary to better determine the properties of this interfacial region, which is also suspected of harbouring some convective overshooting.

6 Some results of the ESTER project

A detailed account of the present achievements of the ESTER project may be found in the two papers by Espinosa Lara & Rieutord (2013) and Rieutord & Espinosa Lara (2012). Here, we wish to give a brief summary of the performance and first results

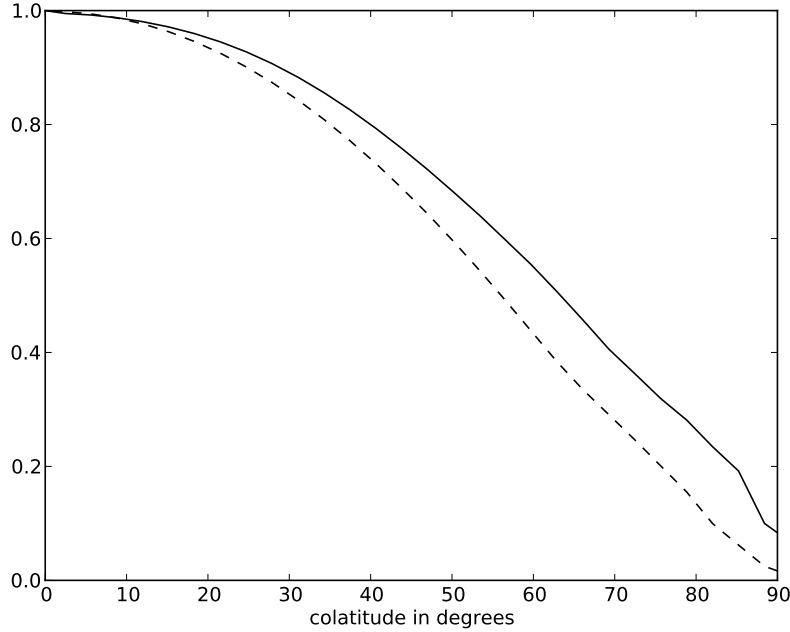


Fig. 2. Variations of the flux with colatitude for a ZAMS star of $30 M_{\odot}$ rotating at 98% of its critical angular velocity. Solid line: the ratio of the flux to the polar flux as a function of the colatitude. Dashed line: the prediction of the von Zeipel law.

of the ESTER code¹.

6.1 Presentation

The ESTER code is solving the equation of stellar structure in two dimensions (2.1), assuming an isentropic convective core and a radiative envelope, thus restricted to the modeling of an early-type star. We use OPAL tables for the equation of state and opacities. Convection in layers is not computed yet : the temperature gradient is assumed to be close to the radiative one, which is fine for stars with mass larger than $1.8 M_{\odot}$.

The discretization of the PDE is based on a spectral decomposition: Chebyshev polynomials radially and spherical harmonics horizontally. The distorted shape of the star is managed through a change of variables mapping the star to the spherical geometry. Internal precision is monitored through spectral conver-

¹The ESTER code is a public domain code that is freely available at <http://code.google.com/p/ester-project/>

Table 1. Comparison between observationally derived parameters of the stars α Oph, α Lyr and α Leo and ESTER models. Data are respectively from Monnier et al. (2010), Monnier et al. (2012) and Che et al. (2011). Note the good matching of the models.

Star	Ras Alhague (α Oph)		Vega (α Lyr)		Regulus (α Leo)	
	Obs.	Model	Obs.	Model	Obs.	Model
Mass (M_{\odot})	$2.4^{+0.23}_{-0.37}$	2.22	$2.15^{+0.10}_{-0.15}$	2.374	4.15 ± 0.06	4.10
R_{eq} (R_{\odot})	2.858 ± 0.015	2.865	2.726 ± 0.006	2.726	4.21 ± 0.07	4.24
R_{pol} (R_{\odot})	2.388 ± 0.013	2.385	2.418 ± 0.012	2.418	3.22 ± 0.05	3.23
T_{eq} (K)	7570 ± 124	7674	8910 ± 130	8973	11010 ± 520	11175
T_{pol} (K)	9384 ± 154	9236	10070 ± 90	10070	14520 ± 690	14567
L (L_{\odot})	31.3 ± 1	31.1	47.2 ± 2	48.0	341 ± 27	351
V_{eq} (km/s)	240 ± 12	242	197 ± 23	205	336 ± 24	335
P_{eq} (days)		0.598		0.672		0.641
P_{pol} (days)		0.616		0.697		0.658
$X_{\text{env.}}$		0.70		0.7546		0.70
$X_{\text{core}}/X_{\text{env.}}$		0.37		0.271		0.5
Z		0.02		0.0093		0.02

gence, the virial test and the energy test (see Rieutord & Espinosa Lara 2012, for details). Iterations follow the Newton algorithm. Fig. 1 illustrates the result of a two-dimensional model for a massive star rotating at 98% of the critical angular velocity. We show here the differential rotation as a function of radius and latitude.

6.2 Gravity darkening law

One of the first results of steady models has been the prediction of the gravity darkening law. Until now, the standard recipe was the von Zeipel law stating that the effective temperature is proportional to the $\frac{1}{4}$ power of the effective gravity. Fitting brightness distributions, interferometric data of rapidly rotating stars have shown that other laws like $T_{\text{eff}} \propto g_{\text{eff}}^{\beta}$ with adjusted β , were more appropriate. This has also been the conclusion of ESTER models with predictions on the β value, although the models show that a power law is not exactly representing the gravity darkening (Espinosa Lara & Rieutord 2011, 2012). In Fig. 2, we show the dependence of the flux with colatitude for a $30 M_{\odot}$ star rotating close to critical angular velocity. The von Zeipel law underestimate the flux in the equatorial regions by more than a factor 3.

6.3 Some models of nearby fast rotating stars

Fundamentals parameters of some nearby (famous) rotating stars, derived by interferometry, have been compared successfully to ESTER models as shown by Tab. 1.

Table 2. Comparison between observationally derived parameters of the stars and tentative two-dimensional models. Data from δ Vel are from Mérand et al. (2011), those of Achernar are from Domiciano de Souza et al. (2012). The models compare nicely with observationally constrained data for the two components of δ Vel A (an eclipsing binary) but have difficulties with Achernar.

Star	Delta Velorum Aa		Delta Velorum Ab		Achernar (α Eri)	
	Obs.	Model	Obs.	Model	Obs.	Model
Mass (M_{\odot})	2.43 ± 0.02	2.43	2.27 ± 0.02	2.27		8.20
R_{eq} (R_{\odot})	2.97 ± 0.02	2.95	2.52 ± 0.03	2.52	11.6 ± 0.3	11.5
R_{pol} (R_{\odot})	2.79 ± 0.04	2.77	2.37 ± 0.02	2.36	8.0 ± 0.4	7.9
T_{eq} (K)	9450	9440	9560	9477	9955^{+1115}_{-2339}	11250
T_{pol} (K)	10100	10044	10120	10115	18013^{+141}_{-171}	16800
L (L_{\odot})	67 ± 3	65.2	51 ± 2	48.5	4500 ± 300	3700
V_{eq} (km/s)	143	143	150	153	298 ± 9	339
P_{eq} (days)		1.045		0.832		1.72
P_{pol} (days)		1.084		0.924		1.68
$X_{\text{env.}}$		0.70		0.70		0.74
$X_{\text{core}}/X_{\text{env.}}$		0.10		0.30		0.05
Z		0.011		0.011		0.04

A further comparison is shown in Tab. 2. δ Vel A is an eclipsing binary that has been studied in detail by interferometry (Mérand et al. 2011; Pribulla et al. 2011). Here too, two dimensional models nicely fit the observationally derived parameters of the two stars². In addition they suggest that they are less metallic than the Sun ($Z \simeq 0.011$). We also note the stronger hydrogen depletion in the core of the most massive one as expected from its fastest evolution.

On the other hand the case of Achernar, the closest Be star to the solar system, is more difficult since none of the explored models really fit the parameters derived from interferometry. Fitting the polar and equatorial radii lead to rather extreme composition or mass suggesting that the star may just have left the main sequence (to which we are constrained at the moment). In view of previous results based on spherical models (Vinicius et al. 2006), this is not totally surprising. However, this star remains a challenging case for two-dimensional models.

7 Outlooks

Presently, ESTER two dimensional models are, strictly speaking, models of internal structure of rotating stars. No time evolution is included. They describe main sequence early-type stars, that is for masses larger than $1.8 M_{\odot}$. Evolution

²The tidal distortion is weak, less than 10^{-4} .

along the main sequence can be mimicked by varying the hydrogen mass fraction in the convective core. Clearly, the next important steps are the extension to low mass stars, which means the computation of outer convective envelope, and the inclusion of time evolution. Other two-dimensional models are currently being developed by Deupree and coworkers (Deupree 2011; Deupree et al. 2012), but in these models the differential rotation should be prescribed.

Such internal structure models are useful to interpret the seismic frequencies of rotating stars when perturbative methods fail. Presently, two codes may deal with two dimensional models: TOP by Reese (Reese et al. 2006, 2013) or ACOR by Ouazzani (Ouazzani et al. 2012). Beside asteroseismology, 2D-models are important for interferometry. Indeed, the interpretation of interferometric visibilities requires the adjustment of models that include the centrifugal distortion of the stars as well as the associated gravity darkening (e.g. Domiciano de Souza et al. 2002). Here, a future improvement will be the calculation of two-dimensional models of atmospheres.

Beyond the obvious improvements that have been mentioned above, we see that progresses in the understanding of rotating stars will have to go through a better modeling of the turbulent transport in all places where turbulence develops. This is certainly the most challenging issue of this modeling since we do not have a general theory of turbulence at hands.

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